Early Journal Content on JSTOR, Free to Anyone in the World

This article is one of nearly 500,000 scholarly works digitized and made freely available to everyone in the world by JSTOR.

Known as the Early Journal Content, this set of works include research articles, news, letters, and other writings published in more than 200 of the oldest leading academic journals. The works date from the mid-seventeenth to the early twentieth centuries.

We encourage people to read and share the Early Journal Content openly and to tell others that this resource exists. People may post this content online or redistribute in any way for non-commercial purposes.

Read more about Early Journal Content at http://about.jstor.org/participate-jstor/individuals/early-journal-content.

JSTOR is a digital library of academic journals, books, and primary source objects. JSTOR helps people discover, use, and build upon a wide range of content through a powerful research and teaching platform, and preserves this content for future generations. JSTOR is part of ITHAKA, a not-for-profit organization that also includes Ithaka S+R and Portico. For more information about JSTOR, please contact support@jstor.org.

Some Theorems relating to Groups of Circles and Spheres.[*]

By Wm. Woolsey Johnson.

1. In the earliest of Cayley's published papers, and that which stands first in his collected works—"On a Theorem in the Geometry of Position," Cambridge Math. Journal (1841), Vol. II, p. 267—the theorem for the multiplication of two determinants (then new) was applied to Carnot's problem: To find the relation which exists between the distances of five points in space. The same method is applied to find the relation between the distances of four points in a plane, and the additional relation which exists when the points are on the circumference of a circle. The m^{th} row of the first of the determinants multiplied together is

$$x_m^2 + y_m^2, \quad x_m, \quad y_m, \quad 1,$$

where x, y are rectangular coordinates, and the n^{th} row of the second is

$$1, \quad -2x_n, \quad -2y_n, \quad x_n^2 + y_n^2;$$

so that the corresponding element of the product is

$$(x_m - x_n)^2 + (y_m - y_n)^2,$$

or the square of the distance between the points (x_m, y_m) and (x_n, y_n). Taking the same four points in each of the factor determinants, and adding to the first the fifth row

$$1, \quad 0, \quad 0, \quad 0,$$

and to the second the fifth row

$$0, \quad 0, \quad 0, \quad 1,$$

and to each a fifth column of zeros, the product is zero, and the required general relation is found. Again, if the four points lie on a circle, each factor determinant is zero without the additional row and column and the product is zero, giving the special relation.

---

[*] Some of the results contained in this paper were presented to the British Association at the Leeds meeting in 1890.

2. By an extension of the method the m^{th} row of the first determinant is taken as

$$x_m^2 + y_m^2 - r_m^2, \quad x_m, \quad y_m, \quad 1,$$

where r_m is the radius of a circle whose centre is (x_m, y_m), and

$$1, \quad -2X_n, \quad -2Y_n, \quad X_n^2 + Y_n^2 - R_n^2$$

as the n^{th} row of the other, so that the corresponding element of the result is

$$(x_m - X_n)^2 + (y_m - Y_n)^2 - r_m^2 - R_n^2,$$

which is defined as the relative power of the circles (x_m, y_m, r_m) and (X_n, Y_n, R_n), these circles belonging to two separate groups. Then, 1°, if there be five circles or six spheres in each group, the product or determinant of powers is equal to zero, and 2°, if there be four circles or five spheres in each group, the power determinant is the product of two determinants each of which depends upon one of the groups.

Mr. Lachlan has developed a great variety of geometrical consequences, derived principally from the first of these theorems, in his memoir "On Systems of Circles and Spheres," Phil. Trans. Vol. 177, (1886), p. 481. It is the object of the present paper to point out some other results derivable from the second theorem, and particularly to evaluate the power determinants for groups of smaller numbers of circles and spheres.

3. Let the four circles whose centres are $(x_1, y_1) \ldots . (x_4, y_4)$ and whose radii are $r_1 \ldots . r_4$ be denoted by a, b, c, d, and put

$$\begin{vmatrix} x_1^2 + y_1^2 - r_1^2 & x_1 & y_1 & 1 \\ x_2^2 + y_2^2 - r_2^2 & x_2 & y_2 & 1 \\ x_3^2 + y_3^2 - r_3^2 & x_3 & y_3 & 1 \\ x_4^2 + y_4^2 - r_4^2 & x_4 & y_4 & 1 \end{vmatrix} = \Delta(a, b, c, d); \tag{1}$$

then for another group of circles A, B, C, D we have

$$\begin{vmatrix} 1 & -2X_1 & -2Y_1 & X_1^2 + Y_1^2 - R_1^2 \\ 1 & -2X_2 & -2Y_2 & X_2^2 + Y_2^2 - R_2^2 \\ 1 & -2X_3 & -2Y_3 & X_3^2 + Y_3^2 - R_3^2 \\ 1 & -2X_4 & -2Y_4 & X_4^2 + Y_4^2 - R_4^2 \end{vmatrix} = -4\Delta(A, B, C, D).$$

Denoting the relative power of the circles a and A by (aA), etc., we have by multiplication

$$\begin{vmatrix} (aA) & (aB) & (aC) & (aD) \\ (bA) & (bB) & (bC) & (bD) \\ (cA) & (cB) & (cC) & (cD) \\ (dA) & (dB) & (dC) & (dD) \end{vmatrix} = -4\Delta(a, b, c, d)\Delta(A, B, C, D). \quad (2)$$

4. The relative power of two circles as (aA) is the square of the distance of their centres diminished by the sum of the squares of the radii. When the circles intersect it is $2Rr\cos\omega$, where ω is the internal angle of intersection, and *it vanishes for two circles which cut at right angles.* When one of the circles reduces to a point, it becomes what is usually called the power of the point relatively to the circle and vanishes for a point on the circumference. When both circles reduce to points, it is the square of their distance.

5. In equation (1) the elements in any row may be called the elements of the corresponding circle. For the circle altogether at an infinite distance the first element is infinitely greater than either of the others. Dividing out this infinite element, we may take for the elements of the circle at infinity

$$1, \quad 0, \quad 0, \quad 0.$$

To preserve equation (2), it is then obviously necessary to take unity as the power of any finite circle relatively to infinity, and zero as the power of infinity relatively to itself.

Again, if the circle become the straight line whose equation is

$$x\cos\alpha + y\sin\alpha = p_0,$$

putting ρ for the infinite distance of the centre from the origin, the elements are $\rho^2 - r^2$ or $(\rho + r)p_0$, $\rho\cos\alpha$, $\rho\sin\alpha$, 1, and dividing by the infinite radius of the circle we may take them to be

$$2p_0, \quad \cos\alpha, \quad \sin\alpha, \quad 0.$$

It follows that to preserve equation (2), the relative power of this line and the finite circle (X, Y, R) must be taken as $2p_0 - 2X\cos\alpha - 2Y\sin\alpha$, or twice the perpendicular from the centre of the circle to the line (as is indeed otherwise evident), the side of the line on which p is positive being regarded as the outside of the infinite circle. The power of infinity relative to a straight line must be taken as zero, and the relative power of two lines as $-2\cos(\alpha - \beta)$; or, since the side on which p is positive is taken as the outside, it is twice the cosine of the angle between the insides of the lines, agreeing with the expression $2rR\cos\omega$, which is now to be divided by the product of the radii of both circles.

6. Making the second group of points in equation (2) identical with the first, we have for the symmetrical determinant of mutual powers

$$\begin{vmatrix} -2r_1^2 & (ab) & (ac) & (ad) \\ (ab) & -2r_2^2 & (bc) & (bd) \\ (ac) & (bc) & -2r_3^2 & (cd) \\ (ad) & (bd) & (cd) & -2r_4^2 \end{vmatrix} = -4\Delta^2(a, b, c, d), \qquad (3)$$

in which for (aa) is put its value $-2r_1^2$, etc.

It follows from this equation that the value of $\Delta(a, b, c, d)$ is independent of the position of the origin and axes.

Thus $\Delta(a, b, c, d)$ depends only upon the configuration of the four circles. If one of them, say a, is removed to infinity, Δ becomes infinite, but taking the reduced elements, as in §5, we have

$$\begin{vmatrix} x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \\ x_4 & y_4 & 1 \end{vmatrix} = \Delta(\infty, b, c, d) = \Delta(b, c, d),$$

or double the area of the triangle whose vertices are the centres of b, c and d.

7. The elements of the first column in the determinant Δ, equation (1), are the powers of the origin, O, with respect to the circles a, b, c, d; that is, in the notation adopted, (Oa), (Ob), (Oc) and (Od). Denoting the corresponding minors by α, β, γ, δ we have

$$\alpha(Oa) + \beta(Ob) + \gamma(Oc) + \delta(Od) = \Delta(a, b, c, d) = \text{constant} \qquad (4)$$

for all positions of O. Moreover, since, if we replace the first column of Δ by a column of units, we have

$$\alpha + \beta + \gamma + \delta = 0, \qquad (5)$$

we may, by subtracting $(\alpha + \beta + \gamma + \delta)R^2 = 0$, replace $(Oa) \ldots . (Od)$ by $(Oa) - R^2 \ldots . (Od) - R^2$, the powers of a circle whose centre is O and radius R relative to $a \ldots . d$. Thus equation (4) expresses a linear relation between the powers of any circle with respect to four given circles.

8. If we eliminate α, β, γ, δ and the constant between four equations of the form (4) (in which A, B, C, D respectively are put for O) and equation (5),

we shall have a relation between the powers which appear in equation (2), namely

$$\begin{vmatrix} (aA) & (aB) & (aC) & (aD) & 1 \\ (bA) & (bB) & (bC) & (bD) & 1 \\ (cA) & (cB) & (cC) & (cD) & 1 \\ (dA) & (dB) & (dC) & (dD) & 1 \\ 1 & 1 & 1 & 1 & 0 \end{vmatrix} = 0, \tag{6}$$

the same relation which is obtained for two groups of five circles, the fifth circle in each group being at infinity.

Values of Δ.

9. In equation (4), let O be the circle orthogonal to b, c and d; the equation reduces to

$$\Delta(a, b, c, d) = (Oa).\Delta(b, c, d); \tag{7}$$

that is, the value of the Δ of four circles is *the power of any one of them relatively to the circle which is orthogonal to the other three, multiplied by the double area of the triangle whose vertices are the centres of these three circles.* In (7), $\Delta(b, c, d)$ is positive when the rotary direction bcd is positive.

If b, c, d are fixed circles, O is fixed, and the circles, a, which satisfy $\Delta(a, b, c, d) =$ constant are those which have a fixed power relatively to O, or, what is the same thing, those which cut orthogonally a certain fixed circle concentric with O.

10. It follows from equation (7) that Δ *vanishes* when a is orthogonal to O; that is, *when there exists a circle O which cuts all four circles orthogonally*, or, when the *four circles have a common radical centre.*

If the centres of b, c and d lie in a straight line, the factor $\Delta(b, c, d)$ vanishes, but $\Delta(a, b, c, d)$ does not vanish in this case, for O becomes a straight line, so that (Oa) contains an infinite factor, namely, the radius of the infinite circle O, the cofactor being, as in §5, twice the perpendicular upon the line O from the centre of a. The radius of a does not now enter the expression, so that a may be taken as a point. If b, c and d are also points, we readily derive a new expression for Δ in this case. For O is now the circle circumscribing the triangle bcd, and denoting its radius by R, we have $4R \times$ area $=$ product of sides, therefore

$$\Delta(a, b, c, d) = 2Rp\Delta(b, c, d) = 4Rp \times \text{area} = p.bc.cd.db;$$

that is, *if* b, c, d *are points on a straight line, and* a *is at a distance* p *from it,*

$$\begin{vmatrix} 0 & x_1 & y_1 & 1 \\ x_2^2+y_2^2 & x_2 & y_2 & 1 \\ x_3^2+y_3^2 & x_3 & y_3 & 1 \\ x_4^2+y_4^2 & x_4 & y_4 & 1 \end{vmatrix} = p.bc.cd.db, \tag{8}$$

the first element being omitted because its corresponding minor vanishes by hypothesis.

11. When one or more of the circles a, b, c, d is replaced by ∞ or by a straight line, the elements being modified as in §5, the value of Δ is modified accordingly. Thus, as mentioned in §6, if ∞ is put for a, $\Delta(a, b, c, d)$ becomes $\Delta(b, c, d)$, the double area of the triangle of centres of the circles b, c, d. If, furthermore, one of the three circles becomes a straight line, the modified value of Δ is readily seen to be the projection upon this line of the distance between the centres of the two circles remaining finite.

12. If a becomes a straight line, b, c, and d remaining finite circles, the general value of Δ, equation (7), holds if we modify accordingly the meaning of (Oa) as explained in §5, namely, (Oa) is twice the perpendicular from the centre of O upon a. That is to say, given a straight line and three circles, we have

$$\begin{vmatrix} 2p_0 & \cos\alpha & \sin\alpha & 0 \\ x_1^2+y_1^2-r_1^2 & x_1 & y_1 & 1 \\ x_2^2+y_2^2-r_2^2 & x_2 & y_2 & 1 \\ x_3^2+y_3^2-r_3^2 & x_3 & y_3 & 1 \end{vmatrix} = 2p \begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}, \tag{9}$$

which gives an expression for p the perpendicular upon a given straight line from the centre of the circle orthogonal to three given circles.

If X and Y are the coordinates of the centre of O, we have in the notation employed

$$p = p_0 - X\cos\alpha - Y\sin\alpha,$$

which becomes X when $p_0 = 0$, $\cos\alpha = -1$, and becomes Y when $p_0 = 0$, $\sin\alpha = -1$; hence the coordinates of the centre of the circle orthogonal to $(x_1, y_1, r_1)(x_2, y_2, r_2)(x_3, y_3, r_3)$ are given by

$$\begin{vmatrix} x_1^2+y_1^2-r_1^2 & y_1 & 1 \\ x_2^2+y_2^2-r_2^2 & y_2 & 1 \\ x_3^2+y_3^2-r_3^2 & y_3 & 1 \end{vmatrix} = 2X \begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix},$$

$$\begin{vmatrix} x_1^2+y_1^2-r_1^2 & x_1 & 1 \\ x_2^2+y_2^2-r_2^2 & x_2 & 1 \\ x_3^2+y_3^2-r_3^2 & x_3 & 1 \end{vmatrix} = -2Y \begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}.$$

13. If b as well as a becomes a straight line, the circle orthogonal to b, c, d has its centre on this line b, that is, at the intersection of this line with the radical axis of c and d. Let p denote the perpendicular from this point upon a, and q the projection on b of the distance between the centres of c and d. Then (Oa) becomes $2pR_1$ and $\Delta(b, c, d)$ becomes qR_2, where R_1 and R_2 are the infinite radii divided out in writing the modified form of Δ. Another expression for Δ arises from taking the finite circles to be a and b. Let M be the intersection of the lines c and d, and let A and B be the centres of the circles. Then O is the circle whose centre is at M and whose squared radius is $\overline{MB}^2 - r_2^2$. The power (Oa) is therefore $\overline{MA}^2 - \overline{MB}^2 - r_1^2 + r_2^2$, while $\Delta(b, c, d)$ when divided by the infinite factors becomes $\sin(\beta - \alpha)$. Thus we have the two expressions

$$\begin{vmatrix} x_1^2 + y_1^2 - r_1^2 & x_1 & y_1 & 1 \\ x_2^2 + y_2^2 - r_2^2 & x_2 & y_2 & 1 \\ 2p_0 & \cos\alpha & \sin\alpha & 0 \\ 2p_0' & \cos\beta & \sin\beta & 0 \end{vmatrix} = 2pq = (\overline{MA}^2 - \overline{MB}^2 - r_1^2 + r_2^2)\sin(\beta - \alpha), \quad (10)$$

where q is the projection of the distance between the centres of the circles upon one of the lines, p the perpendicular from the intersection of this line with the radical axis of the circles upon the other line, M the intersection of the lines, and A and B the centres of the circles.

14. When three of the four circles become straight lines, the fourth being finite, the value of $\Delta(a, b, c, d)$ is independent of the latter and becomes twice the product of the perpendicular upon one of the lines from the intersection of the other two into the sine of the angle between these two lines, or the product of the sides of the triangle formed by the three lines divided by twice the square of the radius of its circumscribing circle.

The Power Determinant for Groups of Four Circles.

15. The determinant of powers of two groups of circles is denoted by $\Pi\begin{pmatrix} a, & b, & c, & d \\ A, & B, & C, & D \end{pmatrix}$, then equation (2) is

$$\Pi\begin{pmatrix} a, & b, & c, & d \\ A, & B, & C, & D \end{pmatrix} = -4\Delta(a, b, c, d)\,\Delta(A, B, C, D).$$

We notice in the first place that the value of Π depends only upon the configuration of the two groups respectively, and not upon their relative position, so that, if $abcd$ and $ABCD$ are two given quadrilaterals, the value of $\Pi\begin{pmatrix} a, & b, & c, & d \\ A, & B, & C, & D \end{pmatrix}$ is unchanged when the relative position of the quadrilaterals is altered.

The value of $\Pi\begin{pmatrix} a, & b, & c, & d \\ A, & B, & C, & D \end{pmatrix}$ vanishes when either Δ vanishes; that is, when either group has a common orthogonal circle (or common radical centre).

When the two groups are identical, we have

$$\Pi\begin{pmatrix} a, & b, & c, & d \\ a, & b, & c, & d \end{pmatrix} = -4\Delta^2(a, b, c, d),$$

which is always negative. If a, b, c, d are points, we have, for the general value of the quantity which vanishes when the points are concyclic,

$$\begin{vmatrix} 0 & \overline{ab}^2 & \overline{ac}^2 & \overline{ad}^2 \\ \overline{ab}^2 & 0 & \overline{bc}^2 & \overline{bd}^2 \\ \overline{ac}^2 & \overline{bc}^2 & 0 & \overline{cd}^2 \\ \overline{ad}^2 & \overline{bd}^2 & \overline{cd}^2 & 0 \end{vmatrix} = -4(\overline{Oa}^2 - R^2)^2\Delta^2(b, c, d), \qquad (11)$$

where Δ is the double area of the triangle bcd, O the centre and R the radius of its circumscribing circle.

The Power Determinant for Groups of Three Circles.

16. Now let a, b, c and A, B, C be any two groups of three circles each; take for d the circle orthogonal to A, B, C, and for D that orthogonal to a, b, c. (The centres are the radical centres of the groups and the radii may of course be imaginary.) Then, in equation (2), the powers (aD), (bD), (cD), (dA), (dB), (dC) vanish, and we have

$$(dD)\begin{vmatrix} (aA) & (aB) & (aC) \\ (bA) & (bB) & (bC) \\ (cA) & (cB) & (cC) \end{vmatrix} = -4\Delta(a, b, c, d)\Delta(A, B, C, D);$$

but, by equation (7),

$$\Delta(a, b, c, d) = -(Dd).\Delta(a, b, c), \qquad \Delta(A, B, C, D) = -(dD).\Delta(A, B, C),$$

hence

$$\Pi\begin{pmatrix} a, & b, & c \\ A, & B, & C \end{pmatrix} = \begin{vmatrix} (aA) & (aB) & (aC) \\ (bA) & (bB) & (bC) \\ (cA) & (cB) & (cC) \end{vmatrix} = -4\Delta(a,\, b,\, c)\,\Delta(A,\, B,\, C)\,(dD).^{*} \qquad (12)$$

That is, the power determinant of two groups of three circles each is — 16 *times the product of the areas of the triangles formed by the centres of the groups and the relative power of the circles respectively orthogonal to the two groups.*

It follows that *the power determinant of two groups of three circles each vanishes when the circles respectively orthogonal to the two groups cut each other orthogonally.*

17. If the two groups are identical we have

$$\Pi\begin{pmatrix} a, & b, & c \\ a, & b, & c \end{pmatrix} = \begin{vmatrix} -2r_1^2 & (ab) & (ac) \\ (ab) & -2r_2^2 & (bc) \\ (ac) & (bc) & -2r_3^2 \end{vmatrix} = 8\Delta^2(a,\, b,\, c)\,R^2, \qquad (13)$$

where R is the radius of the circle orthogonal to the group. This expression vanishes when $R = 0$; that is, when the three circles pass through a common point. It does not, however, vanish when $\Delta(a,\, b,\, c) = 0$ (that is, when the centres of the circles are in a straight line), because R is then infinite, unless the circles have a common radical axis, in which case R is indeterminate. Thus $\Pi\begin{pmatrix} a, & b, & c \\ a, & b, & c \end{pmatrix}$ vanishes only when the three circles have one common point, or two common points real or imaginary.

But, when the centres of a, b, c lie in a straight line, the determinant $\Pi\begin{pmatrix} a, & b, & c \\ a, & b, & c \end{pmatrix}$ admits of reduction. Denoting the sides of the triangle opposite the centres of a, b, c by d_1, d_2, d_3 respectively, and supposing $d_1 = d_2 + d_3$, we shall find

$$\Pi\begin{pmatrix} a, & b, & c \\ a, & b, & c \end{pmatrix} = 2\left[d_2(d_3^2 + r_1^2 - r_2^2) + d_3(d_2^2 + r_1^2 - r_3^2)\right]^2. \qquad (14)$$

---

* It is erroneously stated in Mr. Lachlan's memoir that

$$\left[\Pi\begin{pmatrix} 1, & 2, & 3 \\ 4, & 5, & 6 \end{pmatrix}\right]^2 = \Pi\begin{pmatrix} 1, & 2, & 3 \\ 1, & 2, & 3 \end{pmatrix}\Pi\begin{pmatrix} 4, & 5, & 6 \\ 4, & 5, & 6 \end{pmatrix}$$

(Phil. Trans., Vol. 177 (1886), p. 493). This would imply that the power determinant for groups of three circles depended only upon the configuration of the groups and not upon theii relative position. This property belongs only to groups of four circles. See §15.

The quantity in brackets may also be written

$$d_2\,(d_1^2 + r_3^2 - r_2^2) - d_1\,(d_2^2 + r_3^2 - r_1^2),$$

or symmetrically, if $d_1 + d_2 + d_3 = 0$,

$$-\,d_1d_2d_3 - d_1r_1^2 - d_2r_2^2 - d_3r_3^2.$$

18. If, in equation (12), the centres of a, b, c lie in a straight line, so that $\Delta(a, b, c)$ vanishes, $\Pi\begin{pmatrix} a, & b, & c \\ A, & B, & C \end{pmatrix}$ does not vanish, because (dD) is then infinite, its value being $2pR$, where p is the perpendicular from the radical centre of A, B, C upon the straight line, and R the infinite radius of the circle orthogonal to a, b, c. We have then

$$\Pi\begin{pmatrix} a, & b, & c \\ A, & B, & C \end{pmatrix} = -\,8pR\Delta(a, b, c)\,\Delta(A, B, C).$$

Comparing equations (13) and (14) and disregarding sign, we have for the value of $R\Delta(a, b, c)$, when the centres lie in a straight line and their distances are so taken that $d_1 + d_2 + d_3 = 0$,

$$2R\Delta(a, b, c) = d_1d_2d_3 + d_1r_1^2 + d_2r_2^2 + d_3r_3^2. \tag{15}$$

Substituting, we have for this case

$$\Pi\begin{pmatrix} a, & b, & c \\ A, & B, & C \end{pmatrix} = 4p\Delta(A, B, C)(d_1d_2d_3 + d_1r_1^2 + d_2r_2^2 + d_3r_3^2). \tag{16}$$

If the centres of A, B, C are also in a straight line, and ω is the angle between the lines, p becomes infinite and $= \rho\cos\omega$, where ρ is the radius of the orthogonal circle, and we have, for this case,

$$\left.\begin{aligned} \Pi\begin{pmatrix} a, & b, & c \\ A, & B, & C \end{pmatrix} = 2\cos\omega\,(d_1d_2d_3 + d_1r_1^2 + d_2r_2^2 + d_3r_3^2) \\ (D_1D_2D_3 + D_1R_1^2 + D_2R_2^2 + D_3R_3^2). \end{aligned}\right\} \tag{17}$$

19. Let a, b, c be circles passing through a common point, and let A, B, C be points, then D is the common point of a, b, c, and d is the circle passing through A, B, C; hence, if this circle passes through the common point, the power determinant vanishes. If any point on d, say D_1, be joined with the common point D, its powers relatively to a, b and c are proportional to the distances from D_1 to the other intersections of this line with a, b and c respectively. Thus, if four circles

pass through a common point, and three straight lines through this point meet the circles again in A_1, B_1, C_1, D_1, etc., we shall have

$$\begin{vmatrix} D_1A_1 & D_1B_1 & D_1C_1 \\ D_2A_2 & D_2B_2 & D_2C_2 \\ D_3A_3 & D_3B_3 & D_3C_3 \end{vmatrix} = 0. \tag{18}$$

20. When one or more of the circles are replaced by lines, the powers being taken as in §5, the value of Π is modified by the omission of infinite factors and it is easy to make the corresponding modification of the second member of equation (12). Thus if a be a straight line, P_1, P_2, P_3 being the perpendiculars from the centres of A, B and C, the theorem is

$$\begin{vmatrix} 2P_1 & 2P_2 & 2P_3 \\ (bA) & (bB) & (bC) \\ (cA) & (cB) & (cC) \end{vmatrix} = -4q\Delta(A, B, C).(dD), \tag{19}$$

where q is the projection upon the line a of the line joining the centres of b and c, taken with the sign which would be given to $\Delta(a, b, c)$.

Again, if there be a line in each group, we have

$$\begin{vmatrix} 2\cos\omega & 2P_2 & 2P_3 \\ 2p_2 & (bB) & (bC) \\ 2p_3 & (cB) & (cC) \end{vmatrix} = -4Qq(dD), \tag{20}$$

where ω is the angle between the lines on that side of each on which the perpendiculars have like signs.

If there be two lines in one group, say A and B are lines, we have, dropping the factor 4,

$$\begin{vmatrix} p_1 & p_1' & (aC) \\ p_2 & p_2' & (bC) \\ p_3 & p_3' & (cC) \end{vmatrix} = -\Delta(a, b, c)\sin\omega\,(dD), \tag{21}$$

where ω is the angle between the lines and the signs of the perpendiculars are so taken that $\Delta(A, B, C)$ would be positive.

If there be two lines in each group, we have in like manner

$$\begin{vmatrix} 2\cos(aA) & 2\cos(aB) & 2P_1 \\ 2\cos(bA) & 2\cos(bB) & 2P_2 \\ 2p_1 & 2p_2 & (cC) \end{vmatrix} = -4\sin(ab)\sin(AB).(dD), \tag{22}$$

where $\cos(aA)$ is the cosine of the angle between the lines a and A, etc.

If one group, say A, B, C, consists of three straight lines, the theorem is concerned only with the centres of the circles, hence the other group may be taken as points. If R is the radius of each of the infinite circles and α, β, γ the angles at which they intersect, $\Delta(A, B, C) = R(\sin\alpha + \sin\beta + \sin\gamma)$ and $(Dd) = -\rho^2$, where ρ is the infinite radius of the circle orthogonal to A, B, and C, see §5. Now ρ^2 is the power with respect to either of the circles A, B or C of the radical centre, which becomes the centre of the inscribed circle of the triangle formed by A, B and C; this is readily shown to be $2Rr$, r being the radius of the inscribed circle. Substituting and rejecting the infinite factor R^2, we have, on dividing by 8,

$$\begin{vmatrix} p_1 & p_1' & p_1'' \\ p_2 & p_2' & p_2'' \\ p_3 & p_3' & p_3'' \end{vmatrix} = \Delta(a, b, c)\, r(\sin\alpha + \sin\beta + \sin\gamma). \tag{23}$$

The factor $r(\sin\alpha + \sin\beta + \sin\gamma)$ is equivalent to the area of the triangle formed by the lines A, B and C divided by the radius of its circumscribed circle. Thus the theorem is: Given three points and three straight lines, the determinant of the nine perpendiculars is equal to twice the product of the areas of the triangles formed by the points and by the lines divided by the radius of the circle circumscribing the latter. This theorem is also readily derived by the multiplication of two determinants whose values are $\Delta(a, b, c)$ and $\dfrac{\text{area } ABC}{R}$.

Finally, if both groups consist of straight lines, Π vanishes since D and d are both at infinity, so that $(dD) = 0$, the elements of the determinant now being the cosines of angles such that the corresponding differences in the several rows are equal.

The Power Determinant for Pairs of Circles.

21. Now let a, b and A, B be any two pairs of circles; take for C in equation (12) any circle of the system orthogonal to a, b, and for c, any circle of the system orthogonal to A, B. Then $\Pi\begin{pmatrix} a, & b, & c \\ A, & B, & C \end{pmatrix}$ reduces to $(cC)\,\Pi\begin{pmatrix} a, & b \\ A, & B \end{pmatrix}$. Since D is orthogonal to a, b and c, and d to A, B and C, we may state the conditions imposed upon C, D, c and d thus: C and D are two of the system orthogonal to a and b, and c, d are two of the system orthogonal to A and B,

with the condition that c is orthogonal to D, and d to C. Equation (12) then becomes

$$\Pi\begin{pmatrix} a, & b \\ A, & B \end{pmatrix} = \begin{vmatrix} (aA) & (aB) \\ (bA) & (bB) \end{vmatrix} = -4\Delta(a, b, c)\,\Delta(A, B, C)\frac{(dD)}{(cC)}.$$

Let us for the moment denote the centres of the circles by C, D, c and d; then C and D are upon the radical axis of a, b, and c and d upon that of A, B. Let P_c denote the perpendicular from c upon the line of centres of a, b, and P_C the perpendicular from C upon the line of centres of A, B; then the equation becomes

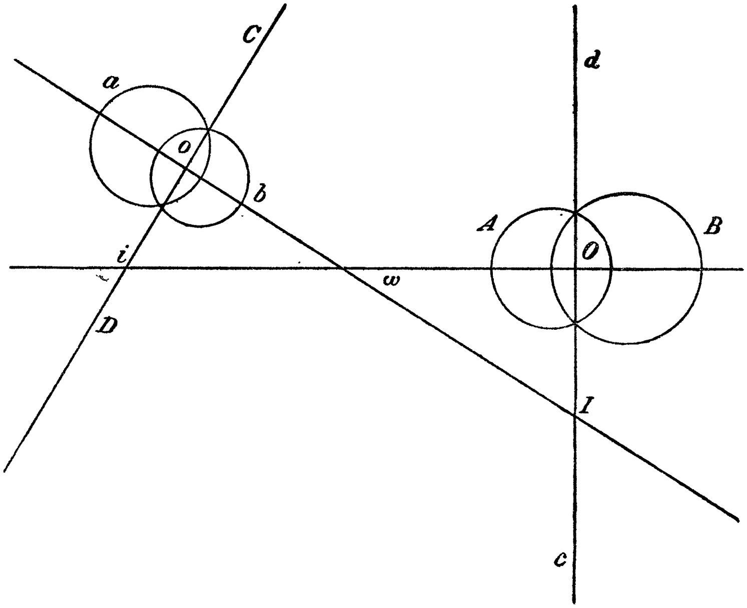
$$\Pi\begin{pmatrix} a, & b \\ A, & B \end{pmatrix} = -4\overline{ab}\,.\,\overline{AB}\,P_c P_C\frac{(dD)}{(cC)}, \tag{24}$$

where $\overline{ab}$ and $\overline{AB}$ are the distances between the centres.

Let the common chord of a, b be $2l$ and o its middle point; let the radical axis of a, b meet the line of centres of A, B in i, and let ω be the angle between the lines of centres.

Now let D in (24) be placed at i; then the circle c becomes the line of centres of A, B, and the point c is at an infinite distance ρ, and we shall have

$$P_c = \rho\cos\omega, \quad (cC) = 2\rho P_C.$$



Substituting in (24), we have

$$\Pi\begin{pmatrix} a, & b \\ A, & B \end{pmatrix} = \begin{vmatrix} (aA) & (aB) \\ (bA) & (bB) \end{vmatrix} = -\, 2\overline{ab}\, \overline{AB} \cos\omega\, (dD). \tag{25}$$

In this equation the point d is any point upon the radical axis of A, B, but (dD) is determinate, for the centre of D is now at i on the radical axis of the system to which c and d belong. Thus

$$(dD) = \overline{iO}^2 + L^2 - (\overline{io}^2 - l^2) = \overline{iI}^2 - \overline{IO}^2 - \overline{io}^2 + L^2 + l^2, \tag{26}$$

If the circles a, b do not intersect, we must of course substitute for l^2, $-r^2$, where r is the distance from o of the limiting points of the coaxial system to which a and b belong.

22. When $\omega = 90°$, the value of Π does not vanish, because (dD) is then infinite. In this case, let the perpendicular from o to the line of centres of A, B be denoted by h, and that from O to the line of centres of a, b by H. Then the limiting value of $(dD) \cos\omega$ is readily shown to be $-\,2hH$; and we have when the lines of centres are at right angles,

$$\Pi\begin{pmatrix} a, & b \\ A, & B \end{pmatrix} = 4\overline{ab}\,.\,\overline{AB}\, hH. \tag{27}$$

In fact, it may be shown that in this case $\dfrac{(dD)}{(cC)}$ in equation (24) becomes -1. In (27) H and h are positive when on the left of the direction a to b, and A to B respectively.

23. When $\overline{ab} = 0$; that is, when the circles a and b are concentric, Π again assumes an indeterminate form. Since ω is now indeterminate, we may employ (27), in which h is now infinite. From the properties of the radical axis we have $\overline{ao}^2 - \overline{bo}^2 = r_1^2 - r_2^2$, whence at the limit $\overline{ab}\,.\,2h = r_1^2 - r_2^2$ and the equation becomes

$$\Pi\begin{pmatrix} a, & b \\ A, & B \end{pmatrix} = 2\,(r_1^2 - r_2^2)\, \overline{AB}\,.\,H. \tag{28}$$

In like manner, when both pairs of circles are concentric, the expression reduces, as it should, to $(r_1^2 - r_2^2)(R_1^2 - R_2^2)$.

24. The value of $\Pi\begin{pmatrix} a, & b \\ A, & B \end{pmatrix}$ vanishes only when (dD) defined by (26) van-

ishes; in other words, when one of the system of circles orthogonal to one pair of circles, say a, b, belongs also to the coaxial system defined by the other pair A, B.

Application to Spheres.

25. The relative power of two spheres is defined in the same manner as that of two circles, so that it is the same as that of the sections made by any plane passing through the centres. It is also the same for the sections made by any plane passing through one of the centres, but not for any plane whatever.

The application to spheres of the methods employed above is so obvious that we shall only mention the general results, which of course require similar modification in the special cases when one or more of the spheres is replaced by a plane or by the sphere at infinity.

Thus, we have for any group of five spheres the determinant $\Delta(a, b, c, d, e)$ whose rows are of the form

$$x^2 + y^2 + z^2 - r^2, \quad x, \quad y, \quad z, \quad 1,$$

(x, y, z and r being in each row the rectangular coordinates of the centre and the radius of one of the spheres), and whose value is independent of the position of the origin and axes. This implies a linear relation between the powers of any sixth sphere relatively to the five given spheres. The value of $\Delta(a, b, c, d, e)$ is the power of one of the spheres relatively to that which cuts the other four orthogonally multiplied by six times the volume of the tetrahedron whose vertices are the centres of these four spheres,

$$\Delta(a, b, c, d, e) = 0$$

is the condition that five spheres shall have a common orthogonal sphere, and

$$\Delta(a, b, c, d, z) = \text{constant}$$

imposes upon the variable sphere z the condition that it shall cut orthogonally a certain sphere whose centre is the radical centre of a, b, c and d.

26. The power determinant of two groups of five spheres, each is

$$\Pi\begin{pmatrix} a, & b, & c, & d, & e \\ A, & B, & C, & D, & E \end{pmatrix} = 8\Delta(a, b, c, d, e)\,\Delta(A, B, C, D, E), \tag{29}$$

which depends only upon the configurations of the two groups, and not upon their relative positions, and which vanishes when either group has a common orthogonal sphere.

The Power Determinant for Groups of Four Spheres.

27. For the power determinant of two groups of four spheres each we derive as in §16,

$$\left.\begin{aligned} \Pi\begin{pmatrix} a, & b, & c, & d \\ A, & B, & C, & D \end{pmatrix} &= \begin{vmatrix} (aA) & (aB) & (aC) & (aD) \\ (bA) & (bB) & (bC) & (bD) \\ (cA) & (cB) & (cC) & (cD) \\ (dA) & (dB) & (dC) & (dD) \end{vmatrix} \\ &= 8\Delta(a, b, c, d)\,\Delta(A, B, C, D).(eE) \end{aligned}\right\} \quad (30)$$

where $\Delta(a, b, c, d)$ now denotes six times the volume of the tetrahedron whose vertices are the centres of a, b, c and d, and e and E are the spheres orthogonal respectively to the two groups.

That is, the power determinant of two groups of four spheres each is 288 *times the product of the volumes of the tetrahedra formed by the centres of the groups and the relative power of the spheres respectively orthogonal to the two groups.*

It follows that this determinant *vanishes when the spheres respectively orthogonal to the two groups cut each other at right angles.*

When the two groups are identical we have

$$\Pi\begin{pmatrix} a, & b, & c, & d \\ a, & b, & c, & d \end{pmatrix} = -16\Delta^2(a, b, c, d)\,R^2, \tag{31}$$

R being the radius of the sphere orthogonal to the group.

The Power Determinant for Groups of Three Spheres.

28. For groups of three spheres we obtain as in §21,

$$\Pi\begin{pmatrix} a, & b, & c \\ A, & B, & C \end{pmatrix} = 8\Delta(a, b, c, d)\,\Delta(A, B, C, D)\frac{(eE)}{(dD)},$$

where d and e are orthogonal to A, B, C, and D, E to a, b, c; D being also orthogonal to e and E to d. Operating as before, this becomes

$$\Pi\begin{pmatrix} a, & b, & c \\ A, & B, & C \end{pmatrix} = 4\Delta(a, b, c)\,\Delta(A, B, C)\cos\omega\,(eE), \tag{32}$$

in which $\Delta(a, b, c)$ and $\Delta(A, B, C)$ are the double areas of the triangles whose vertices are the centres of the spheres of the respective groups, ω is the angle between the planes of centres, and

$$(eE) = \overline{iI}^2 - \overline{io}^2 - \overline{IO}^2 + l^2 + L^2, \tag{33}$$

where the two groups of spheres are supposed to have the common chords $2l$ and $2L$, o and O being their middle points, and where i is the point in which the radical axis of a, b, c meets the plane of centres of A, B, C, I the point in which the radical axis of A, B, C meets the plane of centres of a, b, c.

In equation (32), ω is the angle between those aspects of the planes in which $\Delta(A, B, C)$ and $\Delta(a, b, c)$ have the same sign. When the planes are coincident, $\omega = 180°$ and the equation reduces to equation (12), Io and iO vanishing in the value of (eE).

29. If the three spheres a, b, c do not intersect, we must replace l^2 by $-r^2$, where r is the radius of the circle orthogonal to the three spheres. When this circle is real for each group we have

$$(eE) = \overline{iI}^2 - \overline{io}^2 - \overline{IO}^2 - r^2 - R^2,$$

and this quantity may be defined as the relative power of the two plane circles in question, which are not in the same plane, whose centres are at o, O and whose radii are r, R; i and I being points in each of which the axis of one circle cuts the plane of the other.

From the mode in which (eE) arises, it is obviously the same as the relative power of two spheres passing through the circles and having both their centres in the plane of one of the circles. It vanishes when the circles are such that a sphere can be passed through one of them which shall cut the other circle orthogonally.

The power determinant of two groups of three spheres each vanishes when the circles respectively orthogonal to the groups have this relation.

30. When the planes of the centres of the two groups are perpendicular, the value of Π, when reduced as in §22, becomes

$$\Pi\begin{pmatrix} a, & b, & c \\ A, & B, & C \end{pmatrix} = -8\Delta(a, b, c)\,\Delta(A, B, C)\,hH, \tag{34}$$

where h and H are the distances of o and O from the planes of the centres of A, B, C and a, b, c respectively.

31. When $\Delta(a, b, c) = 0$, that is, when the centres of a, b and c are in a straight line, we may take the plane of centres perpendicular to that of A, B, C, and so employ (34), in which h is now infinite. In fact, if R is the infinite radius of the circle orthogonal to a, b, c, and ϕ the inclination of the line of

centres to the plane of centres of A, B, C, we have $h = R\cos\phi$; hence, employing the notation of §§17 and 18, we have by equation (15), disregarding sign,

$$\Pi\begin{pmatrix} a, & b, & c \\ A, & B, & C \end{pmatrix} = 4\,(d_1d_2d_3 + d_1r_1^2 + d_2r_2^2 + d_3r_3^2)\cos\phi\,\Delta\,(A,\, B,\, C)\,H. \qquad (35)$$

When $\phi = 0$, this reduces to equation (16).

In like manner, if the centres of A, B, C are also in a straight line,

$$\Pi\begin{pmatrix} a, & b, & c \\ A, & B, & C \end{pmatrix} \\ = 2\,(d_1d_2d_3 + d_1r_1^2 + d_2r_2^2 + d_3r_3^2)(D_1D_2D_3 + D_1R_1^2 + D_2R_2^2 + D_3R_3^2)\cos\phi\cos\psi,$$

where ϕ and ψ are the inclinations of the lines of centres of a, b, c and A, B, C to two mutually perpendicular planes passed through the line of centres of A, B, C and a, b, c respectively. Thus, if θ is the angle between two lines parallel to the lines of centres, we have in this case

$$\Pi\begin{pmatrix} a, & b, & c \\ A, & B, & C \end{pmatrix} = 2\,(d_1d_2d_3 + d_1r_1^2 + d_2r_2^2 + d_3r_3^2)(D_1D_2D_3 + D_1R_1^2 + D_2R_2^2 + D_3R_3^2)\cos\theta. \qquad (36)$$

This is identical with equation (17), in which, however, the two lines of centres were supposed to intersect.